Finite-State Markov Modeling of Correlated Rician Fading Channels

Cecilio Pimentel[1]
Dept. of Electronics and Systems
Federal University of Pernambuco
P.O. Box 7800 - 50711-970 - Brazil
cecilio@ufpe.br

T. H. Falk
Queen's University
Electrical and Computer Eng.
K7L 3N6, Kingston, Canada
tiago.falk@ece.queensu.ca

Luciano Lisbôa
Dept. of Electronics and Systems
Federal University of Pernambuco
P.O. Box 7800 - 50711-970 - Brazil
lucianolisboa@bol.com.br

Abstract — **This paper develops a methodology to find Kth-order Markov models that describe the successes and the failures of the transmission of a modulated signal over a time correlated flat fading channel. The order of the Markov model that generates accurate analytical models is estimated for a broad range of fading environments. Fading rates are identified in which the Kth-order Markov model and the Gilbert-Elliott channel model approximate the fading channel with similar accuracy.**

I. Introduction

This paper concerns the development of finite state channel (FSC) models for a discrete communication system composed by an FSK modulator, a time correlated Rician flat fading channel, and a hard quantized non-coherent demodulator. This discrete fading channel model is called the discrete channel with Clarke's autocorrelation (DCCA) model. The FSC model describes the successes and the failures of the symbol transmitted over the fading channel, which is represented mathematically as a binary error sequence.

The main contribution of this paper is to develop a methodology to find accurate Kth-order Markov models for the DCCA model. Markov models with observable states provide simple parameterization and closed-form expressions for some model parameters, e.g. Shannon capacity, are available in the literature. We have applied several statistics to judge model accuracy and to estimate its order. The effect of the signal to noise ratio, fading rate and Rician factor on the accuracy of the proposed models is analyzed. As the fading rate becomes slower, the model may grow to inconvenient sizes. Our second contribution is to identify the system parameters in which the well known two state Gilbert-Elliott channel (GEC) model satisfactorily approximates the DCCA model. The results presented here allow us to accurately study coding performance on correlated fading channels using the analytical techniques developed to analyze burst channels represented as FSC models [1]-[3].

[1]This work received partial support from the Brazilian National Council for Scientific and Technological Development (CNPq) under Grant 300987/96-0.

II. The Channel Model

The probability of an error sequence of length n, $\mathbf{e}_n = e_1 e_2 \dots e_n$, of the DCCA model may be expressed as [4]

$$P(\mathbf{e}_n) = \sum_{l_1=e_1}^{M-1} \cdots \sum_{l_n=e_n}^{M-1} \left(\prod_{k=1}^{n} \frac{\binom{M-1}{l_k}(-1)^{l_k+e_k}}{l_k+1} \right) \frac{\exp\{-\frac{E_s}{N_0} K_R \mathbf{1}^T \mathbf{F}((K_R+1)\,\mathbf{I} + \frac{E_s}{N_0}\overline{\mathbf{C}}\mathbf{F})^{-1}\mathbf{1}\}}{\det(\mathbf{I} + \frac{E_s}{N_0}(1+K_R)^{-1}\overline{\mathbf{C}}\mathbf{F})} \quad (1)$$

where $\overline{\mathbf{C}}$ is the normalized $n \times n$ covariance matrix (the (i,j)th entry of $\overline{\mathbf{C}}$ is $J_0(2\pi f_D (i-j)T)$, where $J_0(x)$ is the zeroth-order Bessel function of the first kind, f_D is the maximum Doppler frequency, T is the symbol interval), $\mathbf{F}$ is a diagonal matrix defined as $\mathbf{F} = \mathrm{diag}(\frac{l_1}{l_1+1}, \dots, \frac{l_n}{l_n+1})$, E_s/N_0 is the signal to noise ratio, $K_R = \eta^2/2\sigma_g^2$ is the Rician factor, $\mathbf{1}$ is a column vector of ones, and the superscript $[\cdot]^T$ indicates the transpose of a matrix. Hereafter, we consider binary modulation so the DCCA model has three parameters K_R, $f_D T$, and E_s/N_0.

A stationary, N-state, FSC model is specified by two $N \times N$ matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$, where the $(i,j)th$ entry of the matrix $\mathbf{P}(e_k)$, $e_k \in \{0,1\}$, is the probability a Markov chain transitions from state $S_{k-1} = i$ to $S_k = j$ and generates an output (error) symbol e_k, that is, $P(S_k = j \mid S_k = i)P(e_k \mid S_k = j)$. The stationary distribution is denoted by $\mathbf{\Pi}$. The probability of an error sequence generated by an FSC model has a matrix form given by

$$P(\mathbf{e}_n) = \mathbf{\Pi}^T \left(\prod_{k=1}^{n} \mathbf{P}(e_k) \right) \mathbf{1}. \quad (2)$$

We consider two classes of FSC models: Kth-order Markov models and the GEC model. Following the ideas introduced in [4], the parameters of each FSC model will be expressed as functions of the probabilities of binary sequences. Then, we apply (1) to estimate these probabilities and to parameterize FSC models that approximates the DCCA correlated fading model.

The Kth-order Markov model can be represented as a function of a first-order Markov chain [5]. Each state of the Kth-order model is represented by a binary string of length K. Given two states $u = u_1 u_2 \cdots u_K$ and $v = v_1 v_2 \cdots v_K$, we say that u and v *overlap progressively* if $u_2 u_3 \cdots u_K = v_1 v_2 \cdots v_{K-1}$. If u and v overlap progressively, then, there is a transition from u to v with probability $P(u_1 v_1 v_2 \cdots v_k)/P(u)$. Otherwise, the state transition probability is zero. Given a state $v = v_1 v_2 \cdots v_K$, $P(e_k \mid v) = v_K$.

The GEC is a two-state FSC model composed of state 0, which produces errors with small probability, namely g, and state 1, where errors occur with higher probability, namely b. The transition probabilities of the Markov chain are $p_{0,1} \triangleq Q$ and $p_{1,0} \triangleq q$. The matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ for the GEC model are given by

$$\mathbf{P}(0) = \begin{bmatrix} (1-Q)\,(1-g) & Q\,(1-b) \\ q\,(1-g) & (1-q)\,(1-b) \end{bmatrix} \quad (3)$$

$$\mathbf{P}(1) = \begin{bmatrix} (1-Q)\,g & Q\,b \\ q\,g & (1-q)\,b \end{bmatrix}. \quad (4)$$

III. Model Evaluation

This section evaluates the accuracy in which the FSC models described in the previous section approximate the DCCA correlated fading model. Motivated by the results presented in [6], we compare next the autocorrelation function (ACF) of the DCCA model with the ACF of FSC models. A closed-form expression for the ACF of the DCCA model is given by

$$R[m] = \frac{(1+K_R)^2\, e^{-2\frac{K_R \frac{E_s}{N_0}}{2+2K_R+(\rho(m)+1)\frac{E_s}{N_0}}}}{\left(2+2K_R+\frac{E_s}{N_0}\right)^2 - \left(\rho(m)\frac{E_s}{N_0}\right)^2} \quad (5)$$

where $\rho(m) = J_0(2\pi m f_D T)$.

The ACF over twenty values of m of the DCCA and the FSC models are compared in Fig. 1. The parameters of the DCCA model are $K_R = 0$, $E_s/N_0 = 15$ dB, $f_D T = 0.05$ (a), $f_D T = 0.02$ (b). Markov models of order up to 6 have been considered. We observe from Fig. 1(a) that the second-order Markov is satisfactory for $f_D T = 0.05$. However, we notice that the ACF of the third-order Markov model is a bit closer to that of the DCCA model, but this strictness may not compensate the doubling of the number of states (we will use other statistics later to confirm this assumption). This tradeoff between accuracy and complexity makes this decision somewhat arbitrary. The ACF's of the third-order Markov and the GEC models are very similar. When $f_D T \lesssim 0.04$, the ACF of the GEC model diverges from the ACF of the DCCA model. This fact is illustrated in Fig. 1(b), where the curve of the fifth-order Markov model approximates better the ACF of the DCCA model than that of the GEC model. Markov models may not be practical for very slowly fading channels ($f_D T < 0.01$) since the number of states grows exponentially with K and large data sizes are necessary to parameterize the model. A similar comparison has been done for $E_s/N_0 = 25$ dB. It was observed that the ACF of the DCCA model decreases more rapidly with m indicating a potential to reduce the order of the Markov approximation. It was also observed that the GEC model becomes accurate for a wider range of $f_D T$ when the signal to noise ratio increases.

We also took into consideration the variational distance between the n-dimensional target measure $P(\mathbf{e}_n)$

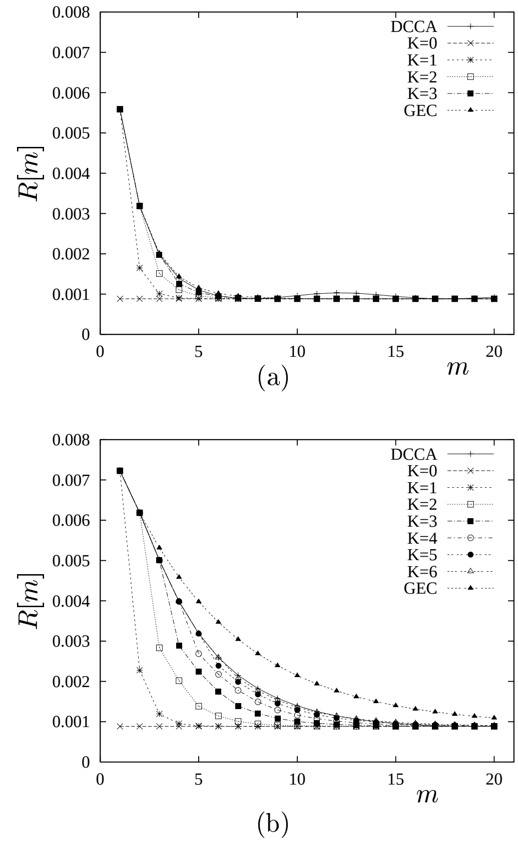

Figure 1: Comparison of the ACFs of the DCCA model, the Kth-order Markov model ($K = 0, 1, \cdots, 6$), and the GEC model. $K_R = 0$, $E_s/N_0 = 15$ dB, and $f_D T = 0.05$ (a), $f_D T = 0.02$ (b).

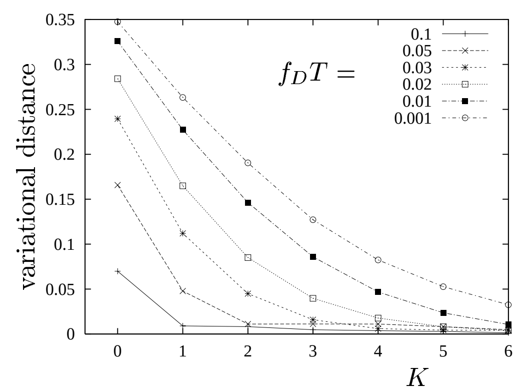

Figure 2: Variational distance versus the order K having $f_D T$ as a parameter. Rayleigh fading ($K_R = 0$), $E_s/N_0 = 15$ dB (a).

Table 1: Order of the Markov model that approximates the DCCA Rayleigh fading model for several values of f_DT. $E_s/N_0 = 10$ dB, 15 dB, 25 dB.

f_DT	K_0 (10 dB)	K_0 (15 dB)	K_0 (25 dB)
0.3	0	0	0
0.2	1	1	0
0.1	2	1	1
0.05	3	2	2
0.04	4	3	2
0.03	5	4	3
0.02	> 6	5	4
0.01	> 7	> 6	5

given by (1) and the measure obtained by the Kth-order Markov model, namely, $P^{(K)}(\mathbf{e}_n)$, which is calculated using (2). The variational distance is defined as $\sum_{\mathbf{e}_n} |P(\mathbf{e}_n) - P^{(K)}(\mathbf{e}_n)|$. Fig. 2 reports the variational distance versus the order K for several values of f_DT, for $E_s/N_0 = 15$ dB. Because of the complexity in obtaining the 2^n possible values of $P(\mathbf{e}_n)$, we have considered $n = 10$. A smaller distance value indicates that the Kth-order Markov model agrees better with the DCCA model. We say that the order of the Markov chain is K_0, when the distance converges to approximately a constant value for $K \geq K_0$. The orders indicated by the convergence of the variational distance, for the range of fading environments investigated, are consistent with those obtained by the ACF method. Table 1 summarizes the choice of K_0 deducted from the curves for selected values of f_DT (the curves used to estimate K_0 for $E_s/N_0 = 10$ dB were not shown for brevity). For the range of signal to noise ratio considered, a memoryless model results for fast fading ($f_DT > 0.3$), while a first-order Markov model is adequate for $f_DT = 0.1$.

IV. Rician Fading

The analysis presented in the previous section is now employed to the DCCA model with Rician fading. ACF curves are plotted in Fig. 3 for Rician fading with $K_R = 5$ dB, $E_s/N_0 = 15$ dB, $f_DT = 0.05$ (a), $f_DT = 0.02$ (b). For small values of m ($m \leq 8$ in Fig. 3(a) and $m \leq 19$ in Fig. 3(b)) the ACF of the DCCA model has a monotonic decreasing exponential behavior that is well approximated by that of FSC models. However, a good fitting is not possible at the oscillatory portion of the AFC curve. A comparison of Fig. 3 with Fig. 1(a),(b), reveals that the Markov models provide a better fit for Rayleigh fading when $E_s/N_0 = 15$ dB. On the other hand, Fig. 4 shows that for $E_s/N_0 = 25$ dB the differences between the ACF of the DCCA and the Markov models are greatly reduced. We observe from Fig. 4 that the GEC model is accurate for $f_DT = 0.02$.

The estimated orders of the Markov model K_0 obtained from the ACF curves and the convergence of the variational distance are shown in Table 2. The values of

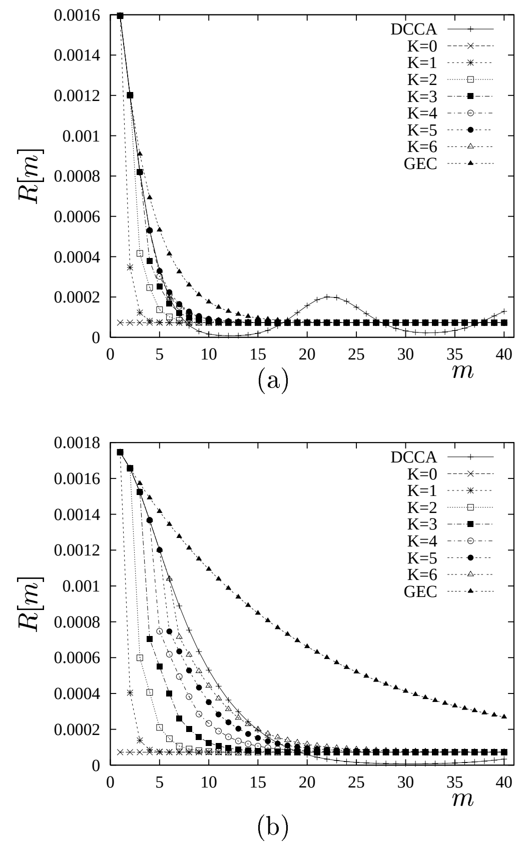

Figure 3: Comparison of the ACFs of the DCCA fading model, the Kth-order Markov model ($K = 0, 1, \cdots, 6$), and the GEC model. $K_R = 5$ dB, $E_s/N_0 = 15$ dB, $f_DT = 0.05$ (a), $f_DT = 0.02$ (b).

K_0 in Table 2 are greater than their correspondings in Table 1. This can be explained by the fact that the mismatch of the ACF curves for Rician fading reflects in the convergence rate of the variational distance. It is worth mentioning that the Markov models indicated in Table 2 for $E_s/N_0 \leq 15$ dB reach good approximation at the first portion of the ACF curve (small m).

V. Equivalence between DCCA and GEC Models

Tables 1 and 2 indicate that the number of states of the Markov models may grow to an inconvenient size for slow fading. It is therefore of interest to evaluate the effectiveness of the GEC model for a wide range of fading parameters. Table 3 classifies the minimum value of f_DT in which the GEC model is approximately statistically equivalent to the DCCA model. Fig. 1(b), 4(b) verify the accuraccy of the GEC model at the lower bound to f_DT shown in Table 3. We found that the GEC model is not adequate when $E_s/N_0 < 15$ dB. Although the GEC model is suitable for fast fading, the zeroth-order and the first-order Markov models are simpler to analyze. This is

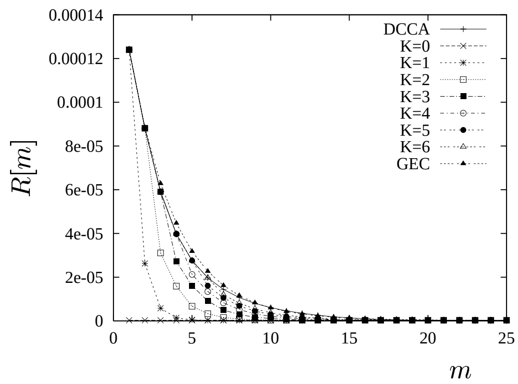

Figure 4: Comparison of the ACFs of the DCCA fading model, the Kth-order Markov model ($K = 0, 1, \cdots, 6$), and the GEC model. $K_R = 5$ dB, $E_s/N_0 = 25$ dB, $f_DT = 0.02$.

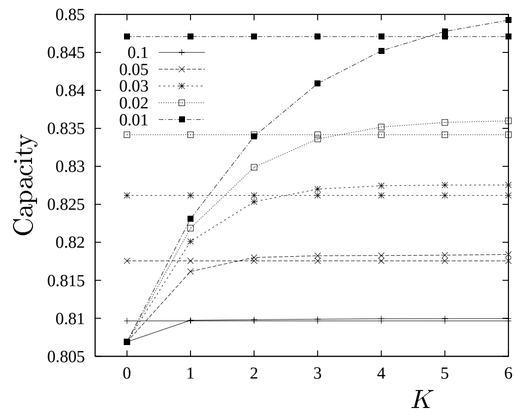

Figure 5: Capacity versus the order K having f_DT as a parameter, for $E_s/N_0 = 15$ dB. Rayleigh fading ($K_R = 0$). $f_DT = 0.1$, 0.05, 0.03, 0.02, 0.01.

Table 2: Order of the Markov model that approximates the DCCA Rician fading model for several values of f_DT. $K_R = 5$ dB. $E_s/N_0 = 10$ dB, 15 dB, 25 dB.

f_DT	K_0 (10 dB)	K_0 (15 dB)	K_0 (25 dB)
0.1	3	2	2
0.05	5	4	2
0.04	6	5	3
0.03	> 6	6	4
0.02	> 6	> 6	5
0.01	> 7	> 6	> 6

the reason of the upper bound $f_DT = 0.1$ in Table 3. This table shows, for example, that for Rayleigh fading, $E_s/N_0 = 25$ dB, $f_DT = 0.01$, the GEC model may be an interesting alternative with respect to the 32-state Markov model indicated in Table 1.

To investigate this equivalence further we will compare the capacity of FSC models. The capacity of the Markov models has closed-form solution, while the capacity of the GEC model was calculated using the algorithm published by Mushkin and Bar-David [7]. In Fig. 5 the capacities are plotted versus the Markov order K, for $E_s/N_0 = 15$ dB. The flat curves correspond to the capacities of the GEC models. For each f_DT, the capacity of the Markov models increases with K and converges to the capacity of the DCCA model. The estimated values of K_0 indicated by the convergence of the capacity curves agree with those shown in Tables 1 and 2. A crossover between the capacity curves reveals the value of K where the Kth-order Markov model and the GEC model have similar a capacities.

Table 3: Range of fading parameters where the GEC model is equivalent to the DCCA model.

E_s/N_0	Rayleigh	Rice ($K_R = 5$ dB)
15 dB	$0.05 \leq f_DT < 0.1$	-
25 dB	$0.01 \leq f_DT < 0.1$	$0.02 \leq f_DT < 0.1$

VI. Conclusions

We have developed FSC models that characterize the error sequence of a communication system operating over a fading channel. Markov models of order up to 6 have been proposed as an approximation to the DCCA model for a broad range of fading environments. It is observed that the first-order approximation is satisfactory for f_DT around 0.1. In the range of signal to noise ratio considered, the Kth-order Markov (for judiciously selected K) is an accurate model for fast and medium fading rates ($f_DT > 0.02$). The GEC model is not adequate for low signal to noise ratio ($E_s/N_0 < 15$ dB), but it becomes accurate, for a broad range of fading rates, when the signal to noise ratio increases. Higher order models are needed to approximate Rician fading with respect to Rayleigh fading with the same fading parameters.

References

[1] L. Wilhelmsson e L. B. Milstein, “On the effect of imperfect interleaving for the Gilbert-Elliott channel”, *IEEE Trans. on Commun.*, vol. 47, pp. 681-688, May 1999.

[2] M. Zorzi, “Some results on error control for burst-error channels under delay constraints,” *IEEE Trans. on Veh. Technol.*, Vol. 50, pp.12-24, Jan. 2001.

[3] C. Pimentel, “Generating series and performance bounds for convolutional codes over burst error channels, ” *IEEE Trans. on Veh. Technol.*, vol. 51, no. 5, pp. 1011-1017, Sept. 2002.

[4] C. Pimentel and I. F. Blake, “Modeling burst channels using partitioned Fritchman’s Markov models,” *IEEE Trans. on Veh. Technol.*, Vol. 47, No. 3, pp.885-899, Aug. 1998.

[5] T. W. Anderson and L. A. Goodman, “Statistical inference about Markov Chains,” *Annals of Mathematical Statistics*, vol. 28, pp. 89-110, Mar. 1957.

[6] C. Tan and N. C. Beaulieu, “On first-order Markov Modeling for the Rayleigh Fading Channel,” *IEEE Trans. on Commun.*, vol. 48, pp. 2032-2040, Dec. 2000.

[7] M. Mushkin and I. Bar-David, “Capacity and coding for the Gilbert-Elliott channels,” *IEEE Trans. on Inform. Theory*, vol. 35, pp. 1277-1289, Nov. 1989.